Stochastic equations for gauge fields

1985 J. Phys. A: Math. Gen. 18 L957

(http://iopscience.iop.org/0305-4470/18/15/010)

J. Phys. A: Math. Gen. 18 (1985) L957-L962. Printed in Great Britain

LETTER TO THE EDITOR

Stochastic equations for gauge fields

Z Haba†

Research Center Bielefeld-Bochum-Stochastics, Bielefeld University, D-4800 Bielefeld 1, FRG

Received 6 August 1985

Abstract. We derive stochastic differential equations for pure Yang-Mills theory in four dimensions and for the three-dimensional Higgs model, which are random perturbations of the instanton equations.

The conventional functional quantisation of gauge theories is plagued by the infrared problems. It appears that for the resolution of these problems a better understanding of some classical aspects of the Yang-Mills theory is needed. We suggest a description of quantum fields in terms of stochastic equations as a tool for a study of these semiclassical problems.

In order to illustrate some aspects of the stochastic description consider first the (imaginary time) quantum mechanics on a Riemannian manifold M. Assume that M is a compact matrix Lie group G. Then, the Brownian motion on the group is described by the equation

$$g^{-1}\,\mathrm{d}g = \mathrm{d}b \tag{1}$$

where b is a matrix from the Lie algebra $L(\mathrm{G})$ of G with matrix elements being independent Wiener processes

$$E(b_{ij}(t)b_{kl}(t')) = \delta_{ik}\delta_{jl}\min(t, t') \tag{2}$$

and dg denotes in this paper the *Stratonovitch* differential (Ikeda and Watanabe 1981).

Let H be a subgroup of G. Consider the fibre bundle π: $\mathrm{G} \to \mathrm{G}/\mathrm{H}$, where G/H is a symmetric space. We can write $g = vh$, with $h \in \mathrm{H}$ and $v \in \mathrm{G}/\mathrm{H}$. Let P_v be the projection on $L(\mathrm{G}) - L(\mathrm{H})$, then the Brownian motion v_t on the coset G/H can be obtained from (1) by means of the projection

$$v^{-1}\,\mathrm{d}v = hP_v\,\mathrm{d}b\,h^{-1} \qquad h^{-1}\,\mathrm{d}h = (1 - P_v)\,\mathrm{d}b. \tag{3}$$

Consider $S^2 = \mathrm{SU}(2)/\mathrm{U}(1)$ as an example. From (3) we can get an equation for the sphere S^2 (Ito 1975)

$$\mathrm{d}n_i = P(n)_{ij}\,\mathrm{d}b_j \tag{4}$$

where

$$P(n)_{ij} = \delta_{ij} - n_i n_{j/n^2} \qquad \text{and} \qquad n^2 = \sum_i n_i n_i.$$

† On leave of absence from Institute of Theoretical Physics, University of Wroclaw, Poland. Supported by Stiftung Volkswagenwerk.

0305-4470/85/150957+06\$02.25

Equation (4) may be considered as an equation for $\boldsymbol{n} \in R^3$, whose solutions stay on a submanifold $S^2 \subset R^3$. In fact, from (4) it follows that $n_i \, dn_i = 0$, hence $n^2 = \text{constant}$, because $P(n)$ projects the vector $d\boldsymbol{b} \in R^3$ onto the tangent space $(TS^2)_n$. Note that $P(n)$ is the metric on the sphere inherited from the flat metric of R^3.

The Euclidean quantum field $\boldsymbol{n}_t(x) \in S^2$ can be treated as a Brownian motion on the manifold of maps $R \to S^2$. We have obtained a stochastic equation for such a field, which can be expressed in the form (Haba 1985a, b)

$$dn^i = p(n)_{ij} \partial_x n^j \, dt + P(n)_{ij} \, db^j \tag{5}$$

where b_t is the Wiener process with values in $L^2(R)$ (see equation (11)), $p_{ij} = \varepsilon_{ijk} n_k / n$ projects onto the tangent space of the sphere ($p^2 = P$) and $\partial_x n_j$ is the Killing vector corresponding to the translational invariance of the L^2 metric. The addition of a Killing vector K to the stochastic equation on M ensures that the solutions still stay on M, and moreover, the expectation values of functionals invariant under the isometry generated by K are independent of K.

Consider now a principal fibre bundle $\pi: P \to M$ with a group G as the fibre. Let $\mathscr{B}$ be the space of (irreducible) connections ω on P and $\mathscr{G}$ an infinite dimensional Lie group of gauge transformations

$$\omega \to \omega^g = g^{-1} \omega g + g^{-1} \, dg. \tag{6}$$

Consider the coset $\mathscr{M} = \mathscr{B}/\mathscr{G}$. Then, $\pi: \mathscr{B} \to \mathscr{M}$ is a principal fibre bundle with the group $\mathscr{G}$ as a fibre (Babelon and Viallet 1979, Singer 1981). $\mathscr{B}$ is an affine space; the tangent space $T\mathscr{B}$ to $\mathscr{B}$ can be identified with the Hilbert space Λ^1 of $L(\mathrm{G})$-valued 1-forms A on M with the scalar product $(A, B) = \mathrm{Tr} \int A \wedge {*B}$. The vertical subspace V_A of $(T\mathscr{B})_A$ is defined as the tangent space to the orbit (6) of $\mathscr{G}$, i.e. $V_A = \{\nabla_A \lambda \, ; \lambda \in \Lambda^0\}$, where ∇_A is the covariant derivative and Λ^0 is the space of $L(\mathrm{G})$-valued functions on M. If we define the horizontal subspace $H_A \subset (T\mathscr{B})_A$ as the orthogonal complement of $T\mathscr{B}$ with respect to the scalar product (,), then this definition of H_A determines the connection form Ω on $\mathscr{B}$, $\Omega = (\nabla_A^* \nabla_A)^{-1} \nabla_A^*$. The connection allows us to identify H_A with $(T\mathscr{M})_{\pi(A)}$ and embed $\mathscr{M}$ as a submanifold in $\mathscr{B}$. In particular, the metric in $\mathscr{B}$ is projected to the metric g in $\mathscr{M}$, i.e. if $X, Y \in (T\mathscr{M})_{\pi(A)}$, then $g_{\pi(A)}(X, Y) = (\tilde{X}, \tilde{Y})$, where $\tilde{X}$ denotes the horizontal lift of X. If $\bar{X}$ and $\bar{Y}$ are arbitrary vectors in $(T\mathscr{B})_A$, then their horizontal parts correspond to vectors X and Y in $(T\mathscr{M})_{\pi(A)}$ with the scalar product

$$g_{\pi(A)}(X, Y) = (\bar{X}, P_A \bar{Y}) \tag{7}$$

where

$$P_A = 1 - \nabla_A (\nabla_A^* \nabla_A)^{-1} \nabla_A^*. \tag{8}$$

We are looking for stochastic equations, whose solutions determine a stochastic process A_t such that the (functional) probability measure corresponding to the process A_t is of the form

$$d\mu \sim [dA] \exp\left(-\tfrac{1}{4} \mathrm{Tr} \int F_{\mu\nu} F_{\mu\nu} \right) \tag{9}$$

where $F_{\mu\nu}$ is the coordinate expression of the curvature of ω. If the measure (9) is to be finite, then the integral $[dA]$ has to be restricted to the manifold $\mathscr{M}$ of orbits, because $\mathrm{Tr}\, F^2$ is constant on the orbit of the gauge group $\mathscr{G}$. For the same reason the process A_t must be defined on $\mathscr{M}$. Note that we have a similar situation in the case of the n field (4), where the Lagrangian is independent of the radial coordinate of $\boldsymbol{n} \in R^3$.

We are going to generalise (3)-(5) to $\mathcal{M}$, i.e. to find a stochastic equation for $A \in \mathcal{B}$, whose solutions stay on a submanifold $\tilde{\mathcal{M}}$ being an embedding of $\mathcal{M}$ in $\mathcal{B}$. If the curve A_t is to be the lift to $\mathcal{B}$ of $\pi(A_t) \in \mathcal{M}$ with respect to the connection $\Omega = (\nabla_A^* \nabla_A)^{-1} \nabla_A^*$, then its tangent $\mathrm{d}A_t/\mathrm{d}t$ must be an element of H_A. Hence, an analogue of (4) has the form

$$\mathrm{d}A_t = P_A\, \mathrm{d}b_t \tag{10}$$

where P_A is defined in (8) and b_t is the Wiener process with values in $L^2(R^3) \times L(\mathrm{G})$

$$E(b_t^a(x) b_{t'}^{a'}(x')) = \delta^{aa'} \delta(x - x') \min(t, t'). \tag{11}$$

Equation (10) can also be expressed in a form analogous to (3) as an equation in the fibre bundle $\pi: \mathcal{B} \to \mathcal{B}/\mathcal{G}$

$$\mathrm{d}\omega^g = g^{-1}\, \mathrm{d}bg \qquad g^{-1}\, \mathrm{d}g = \Omega\, \mathrm{d}b \tag{12}$$

here ω^g (6) represents $\mathcal{B}$ as a manifold of fibres and the action of the connection form Ω on a vector determines an element of $L(\mathcal{G})$, $\Omega\, \mathrm{d}b = (\nabla^* \nabla)^{-1} \nabla_j^*\, \mathrm{d}b_j$.

We may still add to (10) a Killing vector corresponding to an isometry of the metric (7). Let us note that the scalar product (7) is invariant under Euclidean rotations of the vector potential A_k. It is also invariant under a translation of the fibre in the associated fibre bundle (with the adjoint action of $\mathcal{G}$ on the fibre $L(\mathrm{G})$), i.e. under the transformation

$$A(x) \to \exp(\mathrm{i}A_k(x)\Delta x_k) A(x + \Delta x_k) \exp(-\mathrm{i}A_k(x)\Delta x_k).$$

The generator P_k of this isometry has the form $\int \nabla_k A_j \delta/\delta A_j$. Now, the commutator $\Sigma_k [R_k, P_k]$ (R_k is the generator of the rotation around the kth axis) is equal to $\int \varepsilon_{ijk} F_{jk}\, \delta/\delta A_i$. The addition of this Killing vector to (10) leads to the equation

$$\mathrm{d}A_t = {}^*F\, \mathrm{d}t + P_A\, \mathrm{d}b_t \tag{13}$$

where ${}^*F_i = \frac{1}{2}\varepsilon_{ijk} F_{jk}$. For dimensional reasons there should be the square root of the Planck constant $\hbar$ in front of the noise term in (13). Hence, in the limit $\hbar \to 0$ (13) becomes the *instanton equation* in the temporal gauge.

Consider now the functional measure $\mathrm{d}\mu(A)$ corresponding to the solution of (13). Let $\mathrm{d}\mu_0(A)$ be the functional measure corresponding to the solution of (10). Then, according to the Girsanov formula (Ikeda and Watanabe 1981), the measure $\mathrm{d}\mu(A)$ has the form

$$\begin{aligned}\mathrm{d}\mu(A) &= \mathrm{d}\mu_0(A) \exp\left(-\tfrac{1}{2} \int P_A^{-1*} F P_A^{-1*} F\, \mathrm{d}x\, \mathrm{d}t + \int P_A^{-1*} F\, \mathrm{d}b\, \mathrm{d}x\right) \\ &= \mathrm{d}\mu_0(A) \exp\left(-\tfrac{1}{4} \int F_{ij} F_{ij} + Q\right)\end{aligned} \tag{14}$$

where

$$Q = \tfrac{1}{2} \int \varepsilon^{ijk} F_{jk}^a(A)\, \mathrm{d}A_i^a \tag{15}$$

is the topological charge.

In the derivation of (14) we made use of the *horizontality* of ${}^*F$ (i.e. $P_A{}^*F = {}^*F$) and we expressed $P_A\, \mathrm{d}b$ by $\mathrm{d}A$ from (10). Note that because of $\mathrm{Tr}\, \delta^*F/\delta A = 0$ the Ito and the Stratonovitch integrals in (14)-(15) coincide.

In order to find the functional measure $d\mu_0(A)$ corresponding to the solution A_t of (10) we need to derive the short time propagator $p(\Delta t, A, dA')$ describing the Markov process A_t. Then, $d\mu_0(A)$ is a product of the short time propagators. In the finite dimensional case it is known that (Molchanov 1975)

$$p(\Delta t, n, dn') \simeq (2\pi t)^{-1} d\nu(n') \exp[-(n-n')P(n)(n-n')/2\Delta t] \tag{16}$$

where $d\nu(n)$ is the Riemannian volume element on the sphere (in spite of the fact that (4) is expressed as an equation for $\boldsymbol{n} \in R^3$). A formal extension of the arguments leading to (16) shows that for the stochastic process (10) $P(n) \to P_A$ and $d\nu(n) \to d\nu(A)$, where $d\nu(A)$ is the Riemannian volume element on $\mathscr{M}$. In the Coulomb gauge ($\partial^k A_k = 0$) we have (Babelon and Viallet 1979)

$$d\nu(A) = dA(\det \nabla_A^* \nabla_A)^{-1/2} \det \partial^* \nabla_A. \tag{17}$$

Equations (16)-(17) are confirmed by lattice calculations. The configuration space of the lattice gauge theory is a cartesian product of groups attached to each bond b of the lattice. Then, the space of orbits $\mathscr{M}$ is $\Pi_b G_b/\mathscr{G}$, where $\mathscr{G}$ is the group of gauge transformations on the lattice. The transition function p on the coset $\mathscr{M}$ can be obtained (Dowker 1972) from the transition function p_G on $\Pi_b G_b$

$$p(t, g, dg') = \int_{\mathscr{G}} dh\, p_G(t, gh, dg'). \tag{18}$$

If we introduce the parametrisation $g = \exp A$ and assume that for small t only small A (and h close to identity) are relevant, then we get from (18) the equations (16)-(17) (with $P(n) \to P_A$).

The functional measure $d\mu$ resulting from (14), (16) and (17) leads to the conventional functional integral in the Coulomb gauge (Faddeev and Popov 1967) with the topological charge (15) added to the Lagrangian. Equation (13) in the temporal gauge (without P_A) has been obtained earlier by Nicolai (1982). We think that the projection P_A of the noise db in (13) is necessary for consistency of the equations and for the fulfilment of the Gauss law. We find that Nicolai's fermionic determinant is absent. This has also been pointed out by Claudson and Halpern (1985). However, their argument is not satisfactory, because it applies to the model of scalar fields (Parisi and Sourlas 1982, Cecotti and Girardello 1983) with a wrong conclusion. We have discussed in detail the problem of fermions and the applicability of the Girsanov formula in our earlier paper (Haba 1985b). Stochastic equations for gauge fields were studied previously by Asorey and Mitter (1981). These authors treat the spatial part of the Lagrangian as a potential and choose coordinates for $\mathscr{M}$. The introduction of coordinates for the Brownian motion on the sphere (equation (4)) was discussed by Ito (1975). We expect that Ito's procedure applied to the Yang-Mills field leads to the Hamiltonian of Christ and Lee (1980).

Equation (13) is a random perturbation of the instanton equation. We believe that geometric techniques developed for instantons will be fruitful in application to the stochastic equations. In Yang's complex coordinates (Yang 1977, Corrigan *et al* 1978) equation (13) takes the form ($\dot{b}(t, \boldsymbol{x}) = (d/dt)b_t(\boldsymbol{x})$)

$$F_{yz} = \eta_2 - i\eta_1 \tag{19}$$

$$\tfrac{1}{2}(F_{y\bar{y}} + F_{z\bar{z}}) = -i\eta_3 \tag{20}$$

where

$$\eta_i = (P_A \dot{b})_i \equiv \dot{b}_i - \nabla_i \lambda.$$

Consider new field variables (g, B'_z) (where $g \in \mathrm{GL}(n, c)$) related to B_i through the complex gauge transformation $(B'_y = 0)$

$$B_y = g^{-1}\partial_y g, \qquad B_z = g^{-1}B'_z g + g^{-1}\partial_z g. \tag{21}$$

Then, equation (19) reads

$$\partial_y B'_z = g(\dot{b}_2 - \mathrm{i}\dot{b}_1)g^{-1} - \partial_y(g^{-1}\lambda g). \tag{22}$$

Inserting the solution of (22) into (20) we notice that the λ dependent terms cancel. So, we get a simple linear perturbation by noise of Yang's equations for the matrix g.

By means of the dimensional reduction (Taubes 1980) we can get from (13) the two-dimensional Abelian Higgs model (with φ^4 interaction) discussed in our earlier papers as well as a non-Abelian Higgs model discussed by Brink *et al* (1977). In three dimensions the dimensional reduction of (13) leads to the equations

$$\begin{aligned} \mathrm{d}A_i &= \varepsilon_{ij}\nabla_j\varphi \,\mathrm{d}t - \nabla_i \mathscr{D}\varphi \,\mathrm{d}b' + (\delta_{ij} - \nabla_i \mathscr{D}\nabla_j^*)\,\mathrm{d}b_j \\ d\varphi &= \varepsilon_{ij}F_{ij}\,\mathrm{d}t + (1 - \varphi\mathscr{D}\varphi)\,\mathrm{d}b' - \varphi\mathscr{D}\nabla_j^*\,\mathrm{d}b_j \end{aligned} \tag{23}$$

where $\mathscr{D} = (\nabla^*\nabla + \varphi\varphi)^{-1}$, b' and b_j are independent Ad $L(\mathrm{G})$-valued Wiener processes and the multiplication by scalar fields is defined by the adjoint action in $L(\mathrm{G})$.

Equations (23) describe a perturbation of the Bogomolnyi equations for monopoles by the horizontal noise (the vertical space is defined by the infinitesimal gauge transformations $\delta A = \nabla_A \lambda$, $\delta\varphi = \varphi\lambda$). The functional measure corresponding to the solution of (23) describes the Higgs model, where fermions (resulting from the Jacobian) interact with gauge and scalar fields. The model (23) is ultraviolet finite (even the Wick counterterms cancel each other), hence more amenable to the constructive approach than the pure Yang–Mills theory in four dimensions.

I wish to thank Professors S Albeverio, Ph Blanchard and L Streit for their invitation and warm hospitality extended to me during my stay at the Research Center Bielefeld-Bochum-Stochastics (BiBoS) and Center for interdisciplinary Research (ZiF) (Bielefeld University).

References

Asorey M and Mitter P K 1981 *Commun. Math. Phys.* **80** 43
Babelon O and Viallet C M 1979 *Phys. Lett.* **85B** 246
Brink L, Scherk J and Schwarz J H 1977 *Nucl. Phys.* B **121** 77
Cecotti S and Girardello L 1983 *Ann. Phys., NY* **145** 81
Christ N H and Lee T D 1980 *Phys. Rev.* D **22** 939
Claudson M and Halpern M B 1985 *Phys. Lett.* **151B** 281
Corrigan E, Fairlie D, Yates R and Goddard P 1978 *Commun. Math. Phys.* **58** 223
Dowker J S 1972 *J. Phys. A: Math. Gen.* **5** 936
Faddeev L D and Popov V N 1967 *Report* Kiev ITP
Haba Z 1985a *J. Phys. A: Math. Gen.* **18** L347
—— 1985b *Preprint, Instantons with noise. I. Equations for two-dimensional models* BiBoS No 18
Ikeda N and Watanabe S 1981 *Stochastic Differential Equations and Diffusion Processes* (Amsterdam: North-Holland)

Ito K 1975 *Appl. Math. Opt.* **1** 374
Molchanov S 1975 *Russ. Math. Surv.* **30** 1
Nicolai H 1982 *Phys. Lett.* **117B** 408
Parisi G and Sourlas N 1982 *Nucl. Phys.* B **206** 321
Singer I M 1981 *Phys. Scr.* **24** 817
Taubes C H 1980 *Commun. Math. Phys.* **75** 207
Yang C N 1977 *Phys. Rev. Lett.* **38** 1377